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Comparison of strapdown inertial navigation algorithm based on rotation vector and dual quaternion

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Abstract For the navigation algorithm of the strapdown inertial navigation system, by comparing to the equations of the dual quaternion and quaternion, the superiority of the attitude algorithm based on dual quaternion over the ones based on rotation vector in accuracy is analyzed in the case of the rotation of navigation frame. By comparing the update algorithm of the gravitational velocity in dual quaternion solution with the compensation algorithm of the harmful acceleration in traditional velocity solution, the accuracy advantage of the gravitational velocity based on dual quaternion is addressed. In view of the idea of the attitude and velocity algorithm based on dual quaternion, an improved navigation algorithm is proposed, which is as much as the rotation vector algorithm in computational complexity. According to this method, the attitude quaternion does not require compensating as the navigation frame rotates. In order to verify the correctness of the theoretical analysis, simulations are carried out utilizing the software, and the simulation results show that the accuracy of the improved algorithm is approximately equal to the dual quaternion algorithm.

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1. Introduction

Since the emergence of the strapdown inertial navigation system (SINS) concept in the late 1950s, over half a century has seen the great strides in development of the SINS theory and applications. Savage^{1,2} indicated the maturity of the traditional navigation solution of the SINS.³ In 2006, Savage⁴ described a unified mathematical framework for the traditional

strapdown navigation algorithm design. In the traditional algorithm, the gyro outputs are used to maintain a digital reference frame, into which the specific force measurements from accelerometer triads are resolved and then double integrated to acquire velocity and position.^{5–7} The rotation and translation motions of the vehicle are described separately. Quaternion and vector are chosen to represent rotation and translation, respectively.^{8–11}

Dual quaternion is the most concise and efficient mathematical tool to represent rotation and translation simultaneously.^{12–15} In 1992, the dual quaternion was first introduced to the solution of the strapdown inertial navigation.¹⁶ Wu et al.¹² represents the principle of the strapdown inertial navigation in terms of the dual quaternion, and it is concluded that the dual quaternion algorithm is superior to the traditional navigation algorithm in the velocity accuracy, which specifically refers to the thrust velocity. The performance of the attitude and gravitational velocity solution based on the dual quaternion

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algorithm is not analyzed. Although the solution precision of the dual quaternion algorithm is higher than any conventional algorithm, the computational complexity of the dual quaternion algorithm is very huge due to the calculation of the dual vector and dual quaternion.

In this paper, it is proved that there is the superiority of the attitude and gravitational velocity algorithm based on dual quaternion compared with that based on rotation vector in accuracy. An improved navigation algorithm is proposed to resolve the conflict between the navigation accuracy and the computational complexity.

2. Comparison of navigation algorithms

2.1. Comparison of attitude algorithms

For the attitude algorithm based on the rotation vector, suppose that $b(k)$ and $n(k)$ denote the body frame and the navigation frame at time t_k , and $b(k+1)$ and $n(k+1)$ characterize the body frame and the navigation frame at time t_{k+1} , respectively. Let a unit quaternion $q_b(h)$ represent the transformation from the frame $b(k)$ to $b(k+1)$, $Q_{nb}(t_k)$ the transformation from the frame $n(k)$ to the frame $b(k)$, $Q_{nb}(t_{k+1})$ the transformation from the frame $n(k+1)$ to the frame $b(k+1)$, and $p(h)$ the transformation from the frame $n(k)$ to the frame $n(k+1)$, where $h = t_{k+1} - t_k$. According to the characteristics of the quaternion, the following equation can be obtained:

$$r^{n(k+1)} = Q_{nb}(t_{k+1}) \otimes r^{b(k+1)} \otimes Q_{nb}^*(t_{k+1}) \quad (1)$$

where r denotes the vector and the superscript expresses the projection in the corresponding frame, the superscript “*” denotes the conjugate, and the symbol “ \otimes ” denotes the quaternion multiplication.

Eq. (1) is equivalent to the following equation:

$$r^{n(k+1)} = (p^*(h) \otimes Q_{nb}(t_k) \otimes q_b(h)) \otimes r^{b(k+1)} \otimes (p^*(h) \otimes Q_{nb}(t_k) \otimes q_b(h))^* \quad (2)$$

Comparing Eq. (1) with Eq. (2), it can be obtained that

$$Q_{nb}(t_{k+1}) = p^*(h) \otimes Q_{nb}(t_k) \otimes q_b(h) \quad (3)$$

In attitude update cycle, the change of the navigation frame is very slow, namely $p(h) \approx [1 \ 0 \ 0 \ 0]$, so Eq. (3) can be simplified as

$$Q_{nb}(t_{k+1}) \approx Q_{nb}(t_k) \otimes q_b(h) \quad (4)$$

where $q_b(h) = \cos(\sigma/2) + (\sigma/\sigma)\sin(\sigma/2)$, σ denotes the equivalent rotation vector, and $\sigma = |\sigma|$. For $t_k \leq t \leq t_{k+1}$, Eq. (4) can be described as

$$Q_{nb}(t) = Q_{nb}(t_k) \otimes q_b(t - t_k) \quad (5)$$

According to the quaternion differential equation and the time derivative of Eq. (5), the following equation can be obtained:

$$\dot{q}_b = \frac{1}{2} q_b \otimes \omega_{nb}^b \quad (6)$$

The differential equation of the rotation vector can be acquired by Eq. (6), namely Bortz equation^{17,18}:

$$\dot{\sigma} = \omega_{nb}^b + \frac{1}{2} \sigma \times \omega_{nb}^b + \frac{1}{\sigma^2} \left[1 - \frac{\sigma \sin \sigma}{2(1 - \cos \sigma)} \right] \sigma \times (\sigma \times \omega_{nb}^b) \quad (7)$$

The quaternion q_b can be updated by Eq. (7) so as to update the attitude quaternion Q_{nb} according to Eq. (4). Owing to the approximation of Eq. (4), the attitude algorithm based on the rotation vector presents errors inevitably, and the attitude quaternion has to compensate the errors caused by the rotation of navigation frame after several attitude update cycles. However, the errors will be magnified when the vehicle moves at high speed, and the effect of the error compensation would decrease significantly.

The principle of the strapdown inertial navigation utilizing the dual quaternion is expressed by three continuous kinematic equations, namely the thrust velocity equation, the gravitational velocity equation and the position equation, where the determination of the vehicle attitude is derived from the thrust velocity kinematic equation.

Suppose that $T(k)$ and $i(k)$ denote the thrust frame and the inertial frame at time t_k , $T(k+1)$ and $i(k+1)$ characterize the thrust frame and the inertial frame at time t_{k+1} , respectively. Let a unit dual quaternion $\hat{q}_T(h)$ represent the transformation from the frame $T(k)$ to $T(k+1)$, $Q_{iT}(t_k)$ the transformation from the frame $i(k)$ to the frame $T(k)$, $Q_{iT}(t_{k+1})$ the transformation from the frame $i(k+1)$ to the frame $T(k+1)$, and $\hat{p}(h)$ the transformation from the frame $i(k)$ to the frame $i(k+1)$, where $h = t_{k+1} - t_k$. According to the characteristics of the dual quaternion, the following equation can be obtained:

$$\hat{r}^{i(k+1)} = \hat{Q}_{iT}(t_{k+1}) \otimes \hat{r}^{T(k+1)} \otimes \hat{Q}_{iT}^*(t_{k+1}) \quad (8)$$

where \hat{r} denotes the dual vector and the superscript expresses the projection in the corresponding frame.

As the inertial frame is stationary, namely $i(k) = i(k+1)$, the following equation can be obtained:

$$\begin{aligned} \hat{r}^{i(k+1)} &= \hat{r}^{i(k)} = \hat{Q}_{iT}(t_k) \otimes \hat{r}^{T(k)} \otimes \hat{Q}_{iT}^*(t_k) \\ &= \hat{Q}_{iT}(t_k) \otimes (\hat{q}_T(h) \otimes \hat{r}^{T(k+1)} \otimes \hat{q}_T^*(h)) \otimes \hat{Q}_{iT}^*(t_k) \end{aligned} \quad (9)$$

Comparing Eq. (8) with Eq. (9), it can be obtained that

$$\hat{Q}_{iT}(t_{k+1}) = \hat{Q}_{iT}(t_k) \otimes \hat{q}_T(h) \quad (10)$$

Suppose that $\hat{q}_T(h) = \cos(\hat{\sigma}/2) + (\hat{\sigma}/\hat{\sigma})\sin(\hat{\sigma}/2)$, where $\hat{\sigma}$ represents the screw vector lying on the screw axis of the magnitude $\hat{\sigma} = (\hat{\sigma} \cdot \hat{\sigma})^{1/2}$. For $t_k \leq t \leq t_{k+1}$, it can be described by the following equation according to Eq. (10):

$$\hat{Q}_{iT}(t) = \hat{Q}_{iT}(t_k) \otimes \hat{q}_T(t - t_k) \quad (11)$$

After derivation calculus to the above equation, the following equation can be obtained:

$$\frac{d\hat{Q}_{iT}(t)}{dt} = \hat{Q}_{iT}(t_k) \otimes \frac{d\hat{q}_T(t - t_k)}{dt} \quad (12)$$

Putting Eq. (11) into the dual quaternion differential equation $d\hat{Q}_{iT}(t)/dt = (\hat{Q}_{iT}(t) \otimes \hat{\omega}_{iT}^T)/2$, the following equation can be obtained:

$$\frac{d\hat{Q}_{iT}(t)}{dt} = \frac{1}{2} \hat{Q}_{iT}(t_k) \otimes \hat{q}_T(t - t_k) \otimes \hat{\omega}_{iT}^T \quad (13)$$

Comparing Eq. (12) with Eq. (13), it can be acquired that

$$\widehat{\mathbf{Q}}_{iT}(t_k) \otimes \frac{d\widehat{\mathbf{q}}_T(t-t_k)}{dt} = \frac{1}{2} \widehat{\mathbf{Q}}_{iT}(t_k) \otimes \widehat{\mathbf{q}}_T(t-t_k) \otimes \widehat{\omega}_{iT}^T \quad (14)$$

Eq. (14) can be further simplified as

$$\dot{\widehat{\mathbf{q}}}_T = \frac{1}{2} \widehat{\mathbf{q}}_T \otimes \widehat{\omega}_{iT}^T \quad (15)$$

According to the principle of transference by Shen et al.¹⁹, the characteristics of the quaternion are completely inherited by the dual quaternion. Using the same method in deriving the differential equation of the rotation vector, the differential equation of the screw vector can be written as

$$\begin{aligned} \dot{\widehat{\sigma}} &= \widehat{\omega}_{iT}^T + \frac{1}{2} \widehat{\sigma} \times \widehat{\omega}_{iT}^T \\ &+ \frac{1}{\widehat{\sigma}^2} \left[1 - \frac{\widehat{\sigma} \sin \widehat{\sigma}}{2(1 - \cos \widehat{\sigma})} \right] \widehat{\sigma} \times (\widehat{\sigma} \times \widehat{\omega}_{iT}^T) \end{aligned} \quad (16)$$

Comparing Eq. (10) with Eqs. (4) and (10) is the exact expression as the inertial frame is stationary. When the vehicle moves at a high speed, the update of the dual quaternion $\widehat{\mathbf{Q}}_{iT}$ does not have errors according to Eq. (10), so the accuracy of the attitude algorithm based on dual quaternion is higher than that based on rotation vector.

2.2. Comparison of velocity algorithms

The traditional velocity algorithm is described as the following equation:

$$\begin{aligned} \mathbf{V}_m^n &= \mathbf{V}_{m-1}^n + \mathbf{C}_{b(m-1)}^{n(m-1)} \int_{t_{m-1}}^{t_m} \mathbf{C}_{b(t)}^{b(m-1)} \mathbf{f}^b dt + \int_{t_{m-1}}^{t_m} [\mathbf{g}^n \\ &- (2\omega_{ie}^n + \omega_{en}^n) \times \mathbf{V}^n] dt \end{aligned} \quad (17)$$

where \mathbf{V}_m^n and \mathbf{V}_{m-1}^n denote the vehicle velocity at time t_m and t_{m-1} respectively, $\mathbf{C}_{b(m-1)}^{n(m-1)}$ represents the attitude matrix at time t_{m-1} , \mathbf{f}^b the specific force, \mathbf{g}^n the gravitational acceleration, ω_{ie}^n the rotational angular velocity of the Earth, and ω_{en}^n the angular rate of the navigation frame rotation relative to the Earth frame.

In Eq. (17), $\int_{t_{m-1}}^{t_m} \mathbf{C}_{b(t)}^{b(m-1)} \mathbf{f}^b dt$ corresponds to the update of the thrust velocity in dual quaternion algorithm, and Wu et al.¹² has proved the superiority of the thrust velocity algorithm compared with the traditional velocity algorithm. The vehicle velocity relative to the Earth frame is directly determined in the navigation frame for the traditional velocity algorithm, so what need to be compensated are not only the acceleration of gravity \mathbf{g}^n but also the harmful acceleration $(2\omega_{ie}^n + \omega_{en}^n) \times \mathbf{V}^n$. For the dual quaternion algorithm, the determination of the gravitational velocity is in the gravitational velocity frame which is aligned with the Earth frame, and the gravitational velocity is relative to the inertial frame. The update of the gravitational velocity can be described as

$$\begin{aligned} \Delta \mathbf{V}^G &= \sigma' \frac{\sin \sigma}{\sigma} + \frac{(\sigma' \cdot \sigma) \sigma}{\sigma^2} \left(1 - \frac{\sin \sigma}{\sigma} \right) + 2\sigma \\ &\times \sigma' \left(\frac{\sin(\sigma/2)}{\sigma} \right)^2 \end{aligned} \quad (18)$$

where $\sigma' = \mathbf{G} \cdot \Delta t$, $\sigma = \omega_{ie}^e \cdot \Delta t$, Δt is the update cycle of the velocity, \mathbf{G} is the gravitational acceleration and it can be determined by the following equation:

$$\mathbf{G} = \mathbf{C}_n^e \mathbf{g}^n + \omega_{ie} \times (\omega_{ie} \times \mathbf{R}) \quad (19)$$

where \mathbf{C}_n^e represents the direction cosine matrix from the navigation frame to the Earth frame, and \mathbf{R} the position vector of the vehicle. By Eqs. (18), (19), it can be obtained that the update of the gravitational velocity only depends on the vehicle position of the previous time, namely longitude, latitude and height. For the traditional velocity algorithm, $\int_{t_{m-1}}^{t_m} [\mathbf{g}^n - (2\omega_{ie}^n + \omega_{en}^n) \times \mathbf{V}^n] dt$ in Eq. (17) depends on not only the previous time position but also the previous time velocity. As the rate of velocity change is far greater than the rate of position change, the gravitational velocity algorithm is superior to the integration algorithm of harmful acceleration in the traditional velocity solution.

3. Improved navigation algorithm

Suppose that $b(k)$ and $i(k)$ denote the body frame and the inertial frame at time t_k , $b(k+1)$ and $i(k+1)$ characterize the body frame and the inertial frame at time t_{k+1} , respectively. Let a unit quaternion $\mathbf{q}_b(h)$ represent the transformation from the frame $b(k)$ to the frame $b(k+1)$, $\mathbf{Q}_{ib}(t_k)$ the transformation from the frame $i(k)$ to the frame $b(k)$, $\mathbf{Q}_{ib}(t_{k+1})$ the transformation from the frame $i(k+1)$ to the frame $b(k+1)$, and $\mathbf{p}(h)$ the transformation from the frame $i(k)$ to the frame $i(k+1)$, where $h = t_{k+1} - t_k$. According to the characteristics of the quaternion, the following equation can be obtained:

$$\mathbf{r}^{i(k+1)} = \mathbf{Q}_{ib}(t_{k+1}) \otimes \mathbf{r}^{b(k+1)} \otimes \mathbf{Q}_{ib}^*(t_{k+1}) \quad (20)$$

where \mathbf{r} denotes the vector and the superscript expresses the projection in the corresponding frame.

Eq. (20) is equivalent to the following equation:

$$\begin{aligned} \mathbf{r}^{i(k+1)} &= (\mathbf{p}^*(h) \otimes \mathbf{Q}_{ib}(t_k) \otimes \mathbf{q}_b(h)) \otimes \mathbf{r}^{b(k+1)} \otimes (\mathbf{p}^*(h) \\ &\otimes \mathbf{Q}_{ib}(t_k) \otimes \mathbf{q}_b(h))^* \end{aligned} \quad (21)$$

Comparing Eq. (20) with Eq. (21), it can be obtained that

$$\mathbf{Q}_{ib}(t_{k+1}) = \mathbf{p}^*(h) \otimes \mathbf{Q}_{ib}(t_k) \otimes \mathbf{q}_b(h) \quad (22)$$

As the inertial frame is stationary, namely $\mathbf{p}(h) = [1 \ 0 \ 0 \ 0]$, the following equations can be obtained:

$$\mathbf{Q}_{ib}(t_{k+1}) = \mathbf{Q}_{ib}(t_k) \otimes \mathbf{q}_b(h) \quad (23)$$

Suppose that $\mathbf{q}_b(h) = \cos(\sigma/2) + (\sigma/\sigma)\sin(\sigma/2)$, where σ denotes the equivalent rotation vector, and $\sigma = |\sigma|$. For $t_k \leq t \leq t_{k+1}$, Eq. (23) can be described as follows:

$$\mathbf{Q}_{ib}(t) = \mathbf{Q}_{ib}(t_k) \otimes \mathbf{q}_b(t-t_k) \quad (24)$$

After derivation calculus to above equation, the following equation can be obtained:

$$\frac{d\mathbf{Q}_{ib}(t)}{dt} = \mathbf{Q}_{ib}(t_k) \otimes \frac{d\mathbf{q}_b(t-t_k)}{dt} \quad (25)$$

Substituting Eq. (24) into the quaternion differential equation $d\mathbf{Q}_{ib}(t)/dt = (\mathbf{Q}_{ib}(t) \otimes \omega_{ib}^b)/2$, the following equation can be obtained:

$$\frac{d\mathbf{Q}_{ib}(t)}{dt} = \frac{1}{2} \mathbf{Q}_{ib}(t_k) \otimes \mathbf{q}_b(t-t_k) \otimes \omega_{ib}^b \quad (26)$$

Comparing Eq. (25) with Eq. (26), it can be obtained that:

$$\mathbf{Q}_{ib}(t_k) \otimes \frac{d\mathbf{q}_b(t-t_k)}{dt} = \frac{1}{2} \mathbf{Q}_{ib}(t_k) \otimes \mathbf{q}_b(t-t_k) \otimes \omega_{ib}^b \quad (27)$$

Eq. (27) can be further simplified as

$$\dot{\mathbf{q}}_b = \frac{1}{2} \mathbf{q}_b \otimes \boldsymbol{\omega}_{ib}^b \quad (28)$$

Using the same method in deriving the differential equation of the rotation vector, it can be written as the following equation:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\omega}_{ib}^b + \frac{1}{2} \boldsymbol{\sigma} \times \boldsymbol{\omega}_{ib}^b + \frac{1}{\sigma^2} \left[1 - \frac{\sigma \sin \sigma}{2(1 - \cos \sigma)} \right] \boldsymbol{\sigma} \times (\boldsymbol{\sigma} \times \boldsymbol{\omega}_{ib}^b) \quad (29)$$

According to Eqs. (23) and (29), the quaternion \mathbf{Q}_{ib} can be updated. As the attitude determination is usually in geographic frame, it needs to acquire the quaternion \mathbf{Q}_{gb} , where \mathbf{Q}_{gb} expresses the transformation from the geographic frame to the body frame. According to the characteristics of the quaternion, the following equation can be obtained:

$$\mathbf{Q}_{ib} = \mathbf{Q}_{ie} \otimes \mathbf{Q}_{eg} \otimes \mathbf{Q}_{gb} \quad (30)$$

where \mathbf{Q}_{ie} indicates the transformation from the inertial frame to the Earth frame, \mathbf{Q}_{eg} the transformation from the Earth frame to the geographic frame.

According to the multiplicative characteristics of the quaternion, Eq. (30) can be transformed into the following expression:

$$\mathbf{Q}_{gb} = \mathbf{Q}_{eg}^* \otimes \mathbf{Q}_{ie}^* \otimes \mathbf{Q}_{ib} \quad (31)$$

Suppose that the Earth frame coincides with the inertial frame at the initial time, the quaternion \mathbf{Q}_{ie} can be obtained as follows:

$$\mathbf{Q}_{ie} = \begin{bmatrix} \cos(\omega_{ie} t) \\ 0 \\ 0 \\ \sin(\omega_{ie} t) \end{bmatrix} \quad (32)$$

where $\omega_{ie} = |\boldsymbol{\omega}_{ie}|$, and t is the navigation time.

The geographic frame is defined with x directing to the east, y directing to the north and z directing upward vertical. According to the relation between the Earth frame and the geographic frame, the quaternion \mathbf{Q}_{eg} can be described as

$$\mathbf{Q}_{eg} = \frac{1}{2} \begin{bmatrix} (\cos \frac{\lambda}{2} - \sin \frac{\lambda}{2})(\cos \frac{\varphi}{2} + \sin \frac{\varphi}{2}) \\ (\cos \frac{\lambda}{2} - \sin \frac{\lambda}{2})(\cos \frac{\varphi}{2} - \sin \frac{\varphi}{2}) \\ (\cos \frac{\lambda}{2} + \sin \frac{\lambda}{2})(\cos \frac{\varphi}{2} - \sin \frac{\varphi}{2}) \\ (\cos \frac{\lambda}{2} + \sin \frac{\lambda}{2})(\cos \frac{\varphi}{2} + \sin \frac{\varphi}{2}) \end{bmatrix} \quad (33)$$

where λ and φ denote the longitude and latitude, respectively.

Taking Eqs. (32) and (33) into Eq. (31), the attitude quaternion \mathbf{Q}_{gb} can be acquired so as to determine the vehicle attitude. Compared with the traditional attitude algorithm, the improved attitude algorithm does not require compensating the errors caused by the rotation of navigation frame when the vehicle moves.

As stated above, the velocity algorithm based on dual quaternion is superior in the accuracy and the determination of gravitational velocity is described as follows:

$$\mathbf{V}_G^i(t + \Delta t) = \mathbf{V}_G^i(t) + \mathbf{Q}_{ie}(t) \otimes \Delta \mathbf{V}^G \otimes \mathbf{Q}_{ie}^*(t) \quad (34)$$

where $\Delta \mathbf{V}^G$ is described as Eq. (18), and $\mathbf{Q}_{ie}(t)$ is stated as Eq. (32).

The thrust velocity is determined by the following equations:

$$\mathbf{V}_T^i(t + \Delta t) = \mathbf{V}_T^i(t) + \mathbf{Q}_{ib}(t) \otimes \Delta \mathbf{V}^T \otimes \mathbf{Q}_{ib}^*(t) \quad (35)$$

$$\Delta \mathbf{V}^T = \boldsymbol{\sigma}' \frac{\sin \sigma}{\sigma} + \frac{(\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma}}{\sigma^2} \left(1 - \frac{\sin \sigma}{\sigma} \right) + 2 \boldsymbol{\sigma} \times \boldsymbol{\sigma}' \left(\frac{\sin(\sigma/2)}{\sigma} \right)^2 \quad (36)$$

where $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \varepsilon \boldsymbol{\sigma}'$ and $\hat{\boldsymbol{\sigma}}$ is the integration of the thrust velocity twist, namely $\boldsymbol{\sigma} = \int_t^{t+\Delta t} \boldsymbol{\omega}_{ib}^b dt$ and $\boldsymbol{\sigma}' = \int_t^{t+\Delta t} \mathbf{f}^b dt$. $\mathbf{Q}_{ib}(t)$ is obtained by Eq. (23).

The velocity calculation based on dual quaternion is relative to inertial frame and projected into it. However, the velocity of vehicle is usually expressed in the geographic frame, which is relative to the Earth frame. It can be transformed in the following equation:

$$\mathbf{V}_e^g = \mathbf{C}_e^g (\mathbf{C}_i^g \mathbf{V}_i^i + \mathbf{R} \times \boldsymbol{\omega}_{ie}) \quad (37)$$

where \mathbf{C}_e^g is the direction cosine matrix from the Earth frame to the geographic frame, \mathbf{C}_i^g the transformation from the inertial frame to the Earth frame, and $\mathbf{V}_i^i = \mathbf{V}_G^i + \mathbf{V}_T^i$.

For the position determination of vehicle, it can be updated in view of the idea of the improved attitude algorithm, namely updating \mathbf{Q}_{ig} by $\boldsymbol{\omega}_{ig}^g$, accordingly getting \mathbf{Q}_{eg} .

As mentioned above, the vehicle velocity relative to inertial frame can be obtained. Suppose that the local geographic frame is employed as the navigation frame, then the angular velocity of the geographic frame relative to the inertial frame, namely $\boldsymbol{\omega}_{ig}$, can be obtained by the following equation:

$$\boldsymbol{\omega}_{ig}^g = \begin{bmatrix} -\frac{V_{igv}^g}{r_m + H} \\ \frac{V_{ign}^g}{r_n + H} \\ \frac{V_{igx}^g \tan \varphi}{r_n + H} \end{bmatrix} \quad (38)$$

where $\mathbf{V}_{ig}^g = \mathbf{C}_e^g \mathbf{C}_i^g \mathbf{V}_i^i$, r_m denotes the curvature radius of the meridian, r_n the curvature radius of the prime vertical, and H the height of the vehicle.

Suppose that $g(k)$ and $i(k)$ denote the geographic frame and the inertial frame at time t_k , and $g(k+1)$ and $i(k+1)$ characterize the geographic frame and the inertial frame at time t_{k+1} respectively. Let a unit quaternion $\mathbf{q}_g(h)$ the transformation from the frame $g(k)$ to $g(k+1)$, $\mathbf{Q}_{ig}(t_k)$ the transformation from the frame $i(k)$ to the frame $g(k)$, $\mathbf{Q}_{ig}(t_{k+1})$ the transformation from the frame $i(k+1)$ to the frame $g(k+1)$, and $\mathbf{p}(h)$ the transformation from the frame $i(k)$ to the frame $i(k+1)$, where $h = t_{k+1} - t_k$. As the inertial frame is stationary, namely $\mathbf{p}(h) = [1 \ 0 \ 0 \ 0]$, so the following equation can be obtained:

$$\mathbf{Q}_{ig}(t_{k+1}) = \mathbf{Q}_{ig}(t_k) \otimes \mathbf{q}_g(h) \quad (39)$$

Suppose $\mathbf{q}_g(h) = \cos(\sigma/2) + (\boldsymbol{\sigma}/\sigma) \sin(\sigma/2)$, where $\boldsymbol{\sigma}$ denotes the equivalent rotation vector, and $\sigma = |\boldsymbol{\sigma}|$. Imitating the derivation of the improved attitude algorithm, the following equation can be obtained:

$$\dot{\mathbf{q}}_g = \frac{1}{2} \mathbf{q}_g \otimes \boldsymbol{\omega}_{ig}^g \quad (40)$$

According to Eqs. (39), (40), the quaternion \mathbf{Q}_{ig} can be updated. As the position determination is in the geographic frame, it needs to acquire the quaternion \mathbf{Q}_{eg} . According to the characteristics of the quaternion, the following equation can be obtained:

$$\mathbf{Q}_{eg} = \mathbf{Q}_{ie}^* \otimes \mathbf{Q}_{ig} \quad (41)$$

where \mathbf{Q}_{ie} can be obtained by Eq. (32). According to Eq. (41), the direction cosine matrix \mathbf{C}_e^g can be obtained so as to determine the position of the vehicle.

The block diagram of the improved navigation algorithm is shown in Fig. 1. The improved navigation algorithm is based

on the quaternion, the structure of which is similar to the traditional navigation algorithm as shown in Fig. 1. Therefore, the computational complexity based on the improved algorithm is as much as that based on the traditional algorithm. However, for the navigation algorithm based on the dual quaternion, it needs to calculate the dual vector and dual quaternion, which is more than twice in the computational complexity compared with the improved navigation algorithm. Especially, the position determination in dual quaternion algorithm needs the iterative calculation²⁰, so the computation of the improved algorithm is much less than that based on the dual quaternion in the computational complexity.

Comparing Eq. (23) with Eqs. (4) and (23) is the exact expression as the inertial frame is stationary, and the attitude solution in the improved navigation algorithm cannot be affected by the rotation of the navigation frame when the vehicle moves at a high speed so as to bring the attitude superiority of the improved algorithm in accuracy. As stated above, the velocity calculation based on the improved algorithm is in the body frame and Earth frame respectively, where the gravitational velocity only depends on the vehicle position of the previous time, rather than that based on the traditional algorithm which depends not only on the previous time position but also on the previous time velocity. So when the velocity of the vehicle changes dramatically, the velocity calculation based on the improved navigation algorithm is superior over that based on the traditional navigation algorithm in accuracy. To sum up, the improved navigation algorithm has higher precision when the vehicle moves at a high speed.

4. Navigation algorithm simulations

In order to verify the correctness of algorithm analysis, simulations are carried out utilizing the software. In the simulations, the following scenario is assumed. At the very start, the vehicle locates on the surface of the Earth at longitude 126.6287° and latitude 45.7328°. The attitude angles of the

vehicle are all 0°, the east velocity is 100 m/s, and the north and vertical velocity are 0 m/s. The geographic frame is adopted as the navigation frame, which is defined with x directing to the east, y directing to the north and z directing upward vertical.

In the simulations, an ideal trace generator is built for testing the navigation algorithm. In order to verify performances of each navigation algorithm, the vehicle trajectory is simulated in hostile condition, namely the vehicle moves at high speed which leads to the rapid rotation of the navigation frame, and the three attitude angles of the vehicle oscillate simultaneously in a wide range, which bring the coning error into the navigation algorithm tremendously. The specific simulation conditions are as follows: the angular velocity of the yaw and pitch is $0.2\pi\sin(0.2\pi t + 0.5\pi)$ rad/s and the angular velocity of the roll is $0.02\pi\sin(0.2\pi t + 0.5\pi)$ rad/s. The east acceleration of the vehicle is $100\sin(0.2\pi t)$ m/s², and the north and vertical acceleration are all 0 m/s². The frequency of the navigation solution is 50 Hz and the simulation time is 8 h. In order to examine the performance of the navigation algorithm, all of the inertial sensors are assumed under ideal conditions. The navigation algorithms are described as follows and the simulation results are shown in Figs. 2–9. Due to the limited space, the attitude and velocity results are presented only.

Simulation 1: The navigation algorithm adopts the traditional rotation vector and the compensation cycle of the navigation frame is 0.02 s. The results of Simulation 1 are shown in Figs. 2 and 3.

Simulation 2: The improved attitude algorithm is employed, and the velocity and position solutions exploit the traditional algorithm. Figs. 4 and 5 show the results of Simulation 2.

Simulation 3: The navigation solution adopts the improved navigation algorithm and the results are shown in Figs. 6 and 7.

Simulation 4: The navigation algorithm based on dual quaternion is utilized and Figs. 8 and 9 show the results.

Comparing Fig. 2 with Fig. 4, due to the approximation of Eq. (4), the traditional attitude algorithm presents errors

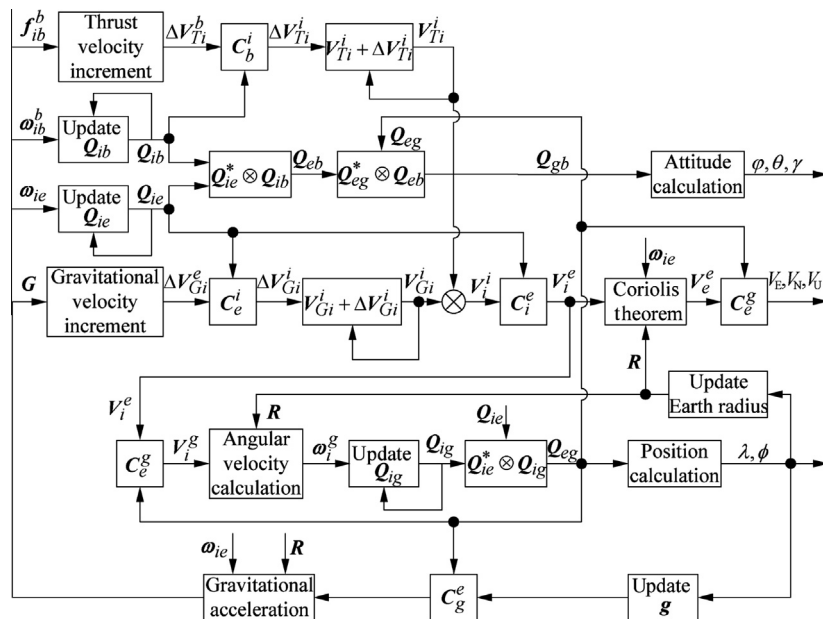


Fig. 1 Block diagram of improved navigation algorithm.

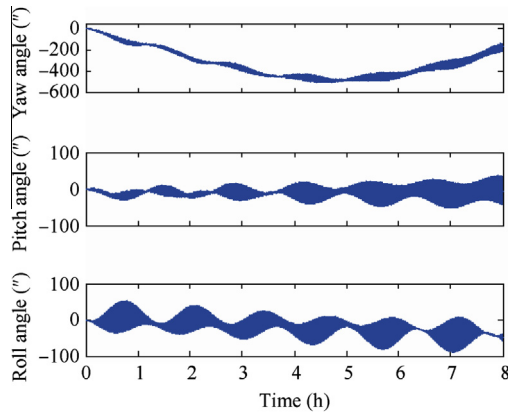


Fig. 2 Curves of attitude error (Simulation 1).

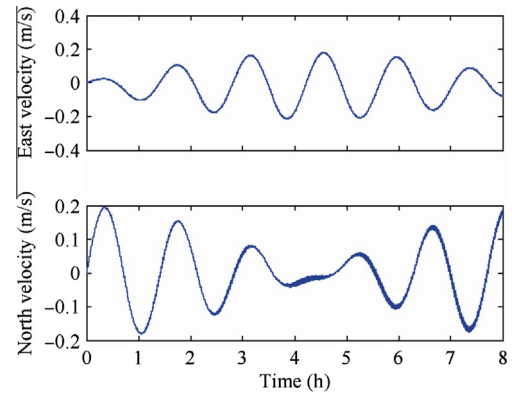


Fig. 5 Curves of velocity error (Simulation 2).

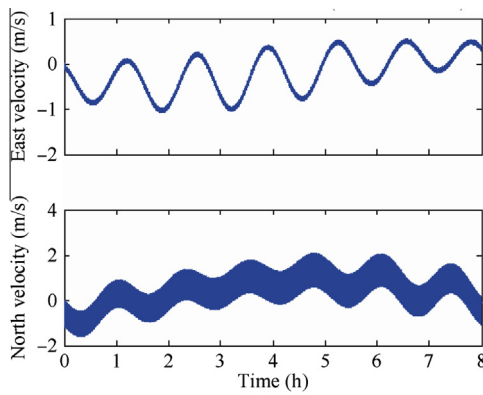


Fig. 3 Curves of velocity error (Simulation 1).

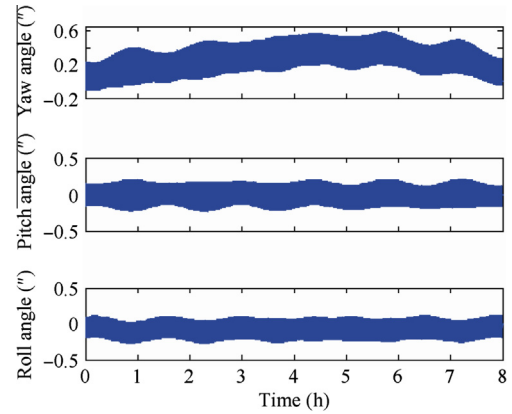


Fig. 6 Curves of attitude error (Simulation 3).

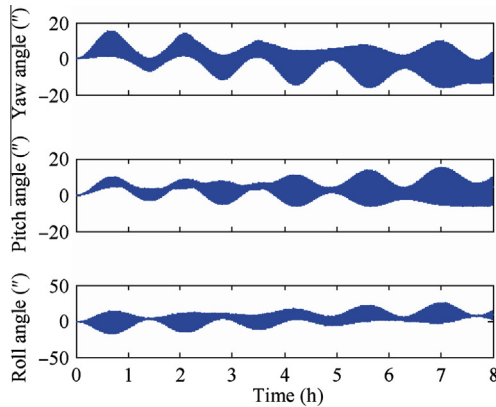


Fig. 4 Curves of attitude error (Simulation 2).

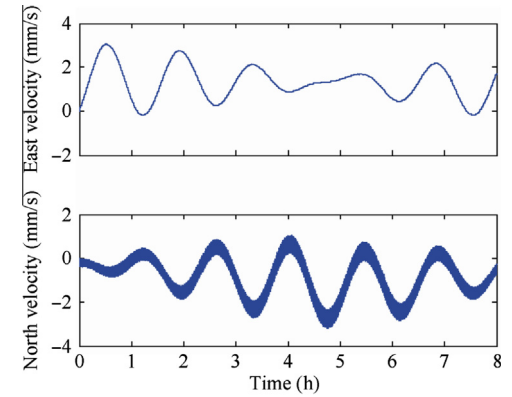


Fig. 7 Curves of velocity error (Simulation 3).

inevitably. Although the attitude errors caused by the rotation of the navigation frame are compensated at very high frequency in the traditional navigation algorithm, the accuracy of the attitude solution is far less than the improved attitude algorithm, so as to verify the attitude superiority of the improved algorithm when the vehicle moves at high speed. Compared the results of Simulation 2 and Simulation 3 which adopt the same attitude algorithm, the navigation precision of Simulation 3 is much better than Simulation 2 as the velocity superiority of

the improved algorithm, which verifies the correctness of the velocity algorithm analysis. From Figs. 6–9, it can be obtained that the navigation solutions based on the improved algorithm and dual quaternion algorithm have the same accuracy which is in the same order of magnitude. However, the computational complexity based on the improved algorithm is as much as that based on the traditional algorithm, which is much less than that based on dual quaternion. As the navigation system is demanding on the real-time performance, the navigation algorithm

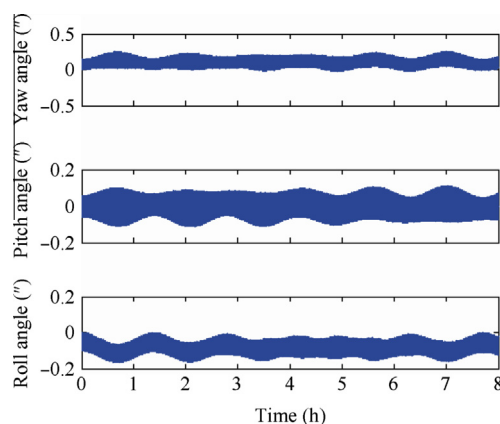


Fig. 8 Curves of attitude error (Simulation 4).

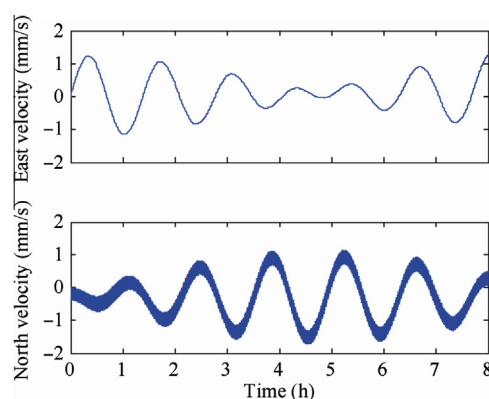


Fig. 9 Curves of velocity error (Simulation 4).

based on the dual quaternion greatly increases the burden on the computer. So the improved navigation algorithm is optimal in the comprehensive properties of the navigation accuracy and computational complexity.

5. Conclusions

The accuracy advantage of the navigation algorithm based on dual quaternion is analyzed and the analytic comparisons indicate that not only the thrust velocity solution but also the attitude and gravitational velocity solution are superior over the traditional navigation algorithm in accuracy. Based on the idea of the dual quaternion algorithm, an improved navigation algorithm is proposed, which is the same as the dual quaternion navigation algorithm in accuracy while as much as the traditional navigation algorithm in computational complexity. A variety of simulations have been carried out to support the analytic conclusions and the numerical results agree well with the analysis.

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